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Weighted estimate of the asymptotic profiles of the Navier–Stokes flow in $\mathbb{R}^n$

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Abstract

We study the asymptotic profiles of the Navier–Stokes flow and the weighted estimate of the spatial and temporal decay rate of the error term. This paper extends the result of Y. Fujigaki and T. Miyakawa in [Y. Fujigaki, T. Miyakawa, Asymptotic profiles of nonstationary incompressible Navier–Stokes flows in the whole space, SIAM J. Math. Anal. 33 (3) (2001) 523–544], where they derived the asymptotic profiles of the Navier–Stokes flow and estimated the temporal decay rate of the error term.

Keywords: Navier–Stokes; Asymptotic profiles

1. Introduction

We consider the Navier–Stokes equations in whole space $\mathbb{R}^n \times \mathbb{R}_+$, $n \geqslant 2$:

$$\operatorname{div} \mathbf{u} = 0, \qquad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p = 0 \tag{1.1}$$

with the initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$.

It is of lasting interest to know the long time behavior of the Navier–Stokes flow. Since a Navier–Stokes flow becomes smooth after long time, we consider usually the strong solutions. By T. Kato [3] and T. Miyakawa [4] it is known that a unique strong solution of the Navier–Stokes equations exists in general L^q-spaces, satisfying

$$\|\mathbf{u}(t)\|_q \leqslant C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}, \quad 1 \leqslant q \leqslant \infty, \tag{1.2}$$

for some C, if $\mathbf{u}_0$ is bounded, integrable, small in L^n, and satisfies $\int (1+|\mathbf{y}|)|\mathbf{u}_0(\mathbf{y})|\,d\mathbf{y} < \infty$.

Later, T. Miyakawa [5] considered the pointwise behavior of the Navier–Stokes flow and showed that there is a small number $C_0 > 0$ so that if

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$$\left|K_t * \mathbf{u}_0(\mathbf{x})\right| \leqslant C_0\left(1+|\mathbf{x}|\right)^{-n-1},$$

then a unique strong solution $\mathbf{u}$ exists global in time, satisfying

$$\left|\mathbf{u}(\mathbf{x},t)\right| \leqslant M_0\left(1+|\mathbf{x}|\right)^{-n-1} \tag{1.3}$$

for some M_0. Here the Gaussian kernel is defined by $K_t(\mathbf{x}) = K(\mathbf{x},t) = (2t)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$. (See also the paper of S. Takahashi [8].)

In [2] Y. Fujigaki and T. Miyakawa have obtained the asymptotic profiles of the solution in L^q space so that if $u_0 \in L^n \cap L^1$ is solenoidal with the properties

$$|\mathbf{y}|^m \mathbf{u}_0(\mathbf{y}) \in L^1, \qquad \left|\mathbf{u}_0(\mathbf{y})\right| \leqslant C\left(1+|\mathbf{y}|\right)^{-n-1},$$
$$u_0 = \operatorname{div}\mathcal{B}, \qquad \mathcal{B} = (b_{jk}), \qquad \left|b_{jk}(\mathbf{y})\right| \leqslant C\left(1+|\mathbf{y}|\right)^{-n}, \quad b_{jk} \in L^1,$$

for some integer m, $1 \leqslant m \leqslant n$, then the global strong solution u satisfies that

$$\begin{aligned}&\Bigg\| u_j(t) - \sum_{1\leqslant|\alpha|\leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\partial_x^\alpha K_t\right)(\cdot)\int \mathbf{y}^\alpha u_{0,j}(\mathbf{y})\,d\mathbf{y} \\ &\quad - \sum_{|\beta|+2p\leqslant m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}\left(\partial_t^p \partial_{\mathbf{x}}^\beta F_{ik}^j\right)(\cdot,t)\int_0^\infty\!\!\int s^p \mathbf{y}^\beta (u_i u_k)(\mathbf{y},s)\,d\mathbf{y}\,ds\Bigg\|_{L^q} \\ &= o\left(t^{-\frac{m}{2}-\frac{n}{2}(1-\frac{1}{q})}\right)\end{aligned} \tag{1.4}$$

as t goes to the infinity. Here the functions F_{jk}^l are defined by $F_{jk}^l(\mathbf{x},t) = \delta_{jk}\partial_{x_l}K_t(\mathbf{x}) + \int_t^\infty \partial_{x_l}\partial_{x_j}\partial_{x_k}K_s(\mathbf{x})\,ds$.

The multi-index symbols in the above statements must be understood as follows: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$, $m-2\alpha = (m-2\alpha_1, m-2\alpha_2, \ldots, m-2\alpha_n)$, $(m-\alpha)! = (m-\alpha_1)!(m-\alpha_2)!\cdots(m-\alpha_n)!$, $\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $\sum_{\beta=0}^{\alpha} = \sum_{\beta_1}^{\alpha_1}\sum_{\beta_2}^{\alpha_2]}\cdots\sum_{\beta_n}^{\alpha_n}$.

In this paper we develop the spatial and temporal decay estimates which extend results of [2,5]. The following theorems refine the known integral norm estimates from a point of view of weighted norm and temporal bound.

Theorem 1.1. *Let $1 \leqslant q \leqslant \infty$. Suppose that $\operatorname{div}\mathbf{u}_0 = 0$, $\mathbf{y}^\alpha \mathbf{u}_0(\mathbf{y}) \in L^1$ for $|\alpha| \leqslant m+1$ for some $m \leqslant n$, and $|\mathbf{y}|^{m+l+1}\mathbf{u}_0(\mathbf{y}) \in L^1$ for some l with $0 \leqslant l < n(1-\frac{1}{q})+1$.*

Moreover, suppose that $\mathbf{u}$ is a (global) strong solution of Navier–Stokes equations (1.1) satisfying the properties (1.2) and (1.3).

Then $\mathbf{u}$ satisfies that

$$\begin{aligned}&\Bigg\| |\cdot|^l \Bigg(u_j(\cdot,t) - \sum_{1\leqslant|\alpha|\leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\partial_x^\alpha K_t\right)(\cdot)\int \mathbf{y}^\alpha u_{0,j}(\mathbf{y})\,d\mathbf{y} \\ &\quad - \sum_{|\beta|+2p\leqslant m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}\left(\partial_t^p \partial_{\mathbf{x}}^\beta F_{ik}^j\right)(\cdot,t)\int_0^\infty\!\!\int s^p y^\beta (u_i u_k)(\mathbf{y},s)\,d\mathbf{y}\,ds\Bigg)\Bigg\|_{L^q} \\ &= \begin{cases} O(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}), & \text{if } m < n, \\ O(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\log t), & \text{if } m = n\end{cases}\end{aligned}$$

as t goes to the infinity.

Remark 1.2. The result of Theorem 1.1 for the case $l = 0$ implies that

$$\Bigg\| u_j(\cdot,t) - \sum_{1\leqslant|\alpha|\leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\partial_x^\alpha K_t\right)(\cdot)\int \mathbf{y}^\alpha u_{0,j}(\mathbf{y})\,d\mathbf{y}$$

$$- \sum_{|\beta|+2p\leqslant m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!} \left(\partial_t^p \partial_{\mathbf{x}}^\beta F_{jk}^l\right)(\cdot,t) \int_0^\infty \int s^p y^\beta (u_l u_k)(\mathbf{y},s)\, d\mathbf{y}\, ds \Bigg\|_{L^q}$$
$$= \begin{cases} O(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}), & \text{if } m<n, \\ O(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\log t), & \text{if } m=n. \end{cases}$$

This estimate can be compared with (1.4) which is a result from [2], and it seems that our result is optimal.

From Theorem 1.1 we have the following reverse estimate for the Navier–Stokes solutions corresponding to special class of initial data.

Corollary 1.3. *Suppose* $\mathbf{u}_0$ *satisfies the same assumption in Theorem* 1.1 *for* $m=1$ *and* $l<n(1-\frac{1}{q})+1$. *In addition, we assume that*

$$\int_{\mathbb{R}^n} \mathbf{y}^\alpha \mathbf{u}_0(\mathbf{y})\, d\mathbf{y} \neq 0 \quad \textit{for some } |\alpha|=1.$$

Then $\mathbf{u}$ *satisfies the reverse estimate*

$$\big\| |\cdot|^l \mathbf{u} \big\|_{L^q} \geqslant C t^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}$$

for large $t>0$.

For the case $l=0$ is already driven by M. Schonbek [7], T. Miyakawa and M. Schonbek [6].

Recently, L. Brandolese and F. Vigneron [1] have derived the spatial and time asymptotic behavior of the Navier–Stokes equations as follows: given $\theta > \frac{n+1}{2}$, let $\mathbf{u}(\mathbf{x},t)$ be a global strong solution of Navier–Stokes equations in $\mathbb{R}^n$ with the property

$$\big|\mathbf{u}(\mathbf{x},t)\big| \leqslant C_0\big(1+|\mathbf{x}|\big)^{-\alpha}(1+t)^{-\frac{\beta}{2}},$$

for any $\alpha,\beta \geqslant 0$ such that $\alpha+\beta\leqslant\theta$, then

$$\mathbf{u}(\mathbf{x},t) = K_t * \mathbf{u}_0(\mathbf{x}) + \nabla\Pi(\mathbf{x},t) + |\mathbf{x}|^{-n-1} E\left(\frac{x}{\sqrt{t+1}};t\right) + \mathcal{R}(\mathbf{x},t),$$

as $|\mathbf{x}|$ goes to the infinity, where E is a given function satisfying an exponential decay

$$\big|E(\mathbf{x},t)\big| \leqslant C e^{-c|\mathbf{x}|^2} \|u\|^2_{L^2([0,t].L^2)}$$

and the remainder $\mathcal{R}$ satisfy that, for any $0\leqslant\alpha\leqslant\min\{1,\theta-\frac{n+1}{2}\}$, and all $t\geqslant 1$,

$$\big|\mathcal{R}(\mathbf{x},t)\big| \leqslant C_\alpha |\mathbf{x}|^{-n-1-\alpha} t^{-\frac{1}{2}+\frac{\alpha}{2}}, \quad \text{if } \theta > \frac{n+3}{2},$$
$$\big|\mathcal{R}(\mathbf{x},t)\big| \leqslant C_\alpha |\mathbf{x}|^{-n-1-\alpha} t^{\frac{n+2+\alpha}{2}-\frac{\alpha}{2}+\epsilon}, \quad \text{if } \frac{n+1}{2} < \theta \leqslant \frac{n+3}{2}.$$

Moreover, Π is a tensor function of the form $\Pi(\mathbf{x},t) = \pi^{-\frac{n}{2}}\Gamma(\frac{n+2}{2})\sum_{h,k}(\frac{\delta_{h,k}}{n|\mathbf{x}|^n} - \frac{x_h x_k}{|\mathbf{x}|^{n+2}})\int_0^t \int (u_h u_k)(\mathbf{y},s)\, d\mathbf{y}\, ds$.

2. Preliminaries

Let us consider the whole domain $\mathbb{R}^n$, $n\geqslant 1$, the associated Gaussian kernel $K_t(\mathbf{x}) = K(\mathbf{x},t) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$. Then $K(\mathbf{x}-\mathbf{y},t)$ can be expressed by the power series in the whole space

$$K_t(\mathbf{x}-\mathbf{y}) = \sum_{\alpha\geqslant 0} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\mathbf{x}}^\alpha K_t(\mathbf{x}) \mathbf{y}^\alpha.$$

Note that the radius of convergence of the above series is infinity.

Lemma 2.1.

$$K(\mathbf{x}-\mathbf{y},t)=\sum_{0\leqslant|\alpha|\leqslant m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K_t(\mathbf{x})\mathbf{y}^{\alpha}+R_m(\mathbf{x},\mathbf{y},t),$$

where

$$R_m(\mathbf{x},\mathbf{y},t)=\sum_{|\alpha|=m+1}\int_0^1\frac{(-1)^{m+1}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K_t(\mathbf{x}-c\mathbf{y})\mathbf{y}^{\alpha}\,dc.$$

Direct computation shows that

$$\left|R_m(\mathbf{x},\mathbf{y},t)\right|\leqslant C(m)t^{-\frac{m+1}{2}-\frac{n}{2}}|\mathbf{y}|^{m+1}\int_0^1 e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,dc.$$

Proof. Let

$$R_m(\mathbf{x},\mathbf{y},t)=K_t(\mathbf{x}-\mathbf{y})-\sum_{0\leqslant|\alpha|\leqslant m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K(\mathbf{x},t)\mathbf{y}^{\alpha}.$$

By Taylor theorem,

$$R_m(\mathbf{x},\mathbf{y},t)=\sum_{|\alpha|=m+1}\int_0^1\frac{(-1)^{m+1}}{\alpha!}\partial_{x}^{\alpha}K_t(\mathbf{x}-c\mathbf{y})\mathbf{y}^{\alpha}\,dc,$$

and have that for $|\alpha|=m+1$

$$\left|\partial_{\mathbf{x}}^{\alpha}K_t(\mathbf{x}-c\mathbf{y})\mathbf{y}^{\alpha}\right|\leqslant t^{-\frac{n}{2}-\frac{m+1}{2}}e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{4t}}.$$

For the last inequality in the above, we used the fact that

$$|\mathbf{x}-c\mathbf{y}|^{m+1}e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{4t}}\leqslant C(m)t^{\frac{m+1}{2}}e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}.\qquad\square$$

Remark 2.2. From the results of Widder [9,10] the terms $\partial_{\mathbf{x}}^{\alpha}K_t(\mathbf{x})$ can be represented by

$$\partial_{\mathbf{x}}^{\alpha}K_t(\mathbf{x})=\omega_{\alpha}(\mathbf{x},t)\frac{(-1)^{|\alpha|}}{2^{|\alpha|}},$$

where

$$\omega_{\alpha}(\mathbf{x},t)=K_t(\mathbf{x})v_{\alpha}(\mathbf{x},-t)t^{-|\alpha|},\qquad v_{\alpha}(\mathbf{x},t)=|\alpha|!\sum_{\beta=0}^{[\frac{\alpha}{2}]}\frac{t^{|\beta|}}{\alpha!}\frac{\mathbf{x}^{\alpha-2\beta}}{(\alpha-2\beta)!}.$$

We note that $v_{\alpha}(\mathbf{x},t)$ is a homogeneous heat polynomial in terms of $\mathbf{x}$ and $t^{\frac{1}{2}}$ of degree $|\alpha|$.

It is known from [9,10] that $\{v_{\alpha}(\mathbf{x},t)\colon |\alpha|=n\}$ consists of a basis of the set of the heat polynomial of degree n in terms of $\mathbf{x}$ and $t^{1/2}$.

Define $F_{lj}^{i}(\mathbf{x},t)=\delta_{li}\partial_{x_j}K_t(\mathbf{x})+\partial_{x_i}\partial_{x_j}\partial_{x_l}N*K_t(\mathbf{x})$. As in the paper of T. Miyakawa [2], we have that

$$F_{lj}^{i}(\mathbf{x},t)=\delta_{lj}\partial_{x_i}K_t(\mathbf{x})+\int_t^{\infty}\partial_{x_i}\partial_{x_l}\partial_{x_j}K_s(\mathbf{x},s)\,ds.$$

Hence it follows that

$$\left|\left(\partial_{\mathbf{x}}^{\alpha}\partial_t^p F_{lj}^i\right)(\mathbf{x},t)\right| \leqslant t^{-\frac{n+1}{2}-\frac{m}{2}} e^{-\frac{|\mathbf{x}|^2}{8t}} + \int_t^{\infty} s^{-\frac{n+3}{2}-\frac{m}{2}} e^{-\frac{|\mathbf{x}|^2}{8s}}\,ds \quad \text{for } |\alpha|+2p=m. \tag{2.1}$$

Here we note that $|\mathbf{x}|^l e^{-\frac{|\mathbf{x}|^2}{4t}} \leqslant C(l) t^{\frac{l}{2}} e^{-\frac{|\mathbf{x}|^2}{8t}}$.

By Taylor theorem we have the following polynomial expansion for $F_{lj}^i(\mathbf{x}-\mathbf{y}, t-s)$.

Lemma 2.3.

$$F_{lj}^i(\mathbf{x}-\mathbf{y}, t-s) = \sum_{0\leqslant|\alpha|+2p\leqslant m} \frac{(-1)^{|\alpha|+2p}}{\alpha!p!}\left(\partial_x^{\alpha}\partial_t^p F_{lj}^i\right)(\mathbf{x},t)\mathbf{y}^{\alpha}s^p + R_{milj}(\mathbf{x},\mathbf{y},t,s),$$

where $R_{m,lj,i}$ is defined by

$$R_{milj}(\mathbf{x},\mathbf{y},t,s) = \sum_{|\alpha|+2p=m+1} \frac{(-1)^{m+1}}{\alpha!p!}\mathbf{y}^{\alpha}s^p \int_0^1\int_0^1 \left(\partial_x^{\alpha}\partial_t^p F_{lj}^i\right)(\mathbf{x}-c_1\mathbf{y}, t-c_2 s)\,dc_1\,dc_2.$$

Direct computation yields that

$$\begin{aligned} &R_{milj}(\mathbf{x},\mathbf{y},t,s) \\ &\quad\leqslant c(m)\int_0^1\int_0^1\left[(t-c_2s)^{-\frac{n+m+2}{2}} e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{4(t-c_2s)}} + \int_{t-c_2s}^{\infty} \tau^{-\frac{n+m+4}{2}} e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{4\tau}}\,d\tau\right]dc_1\,dc_2 \cdot \sum_{|\alpha|+2p=m+1}|\mathbf{y}|^{\alpha}s^p. \end{aligned}$$

3. Asymptotic profiles of solutions of the heat equation and the Stokes equations

Let us consider a heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \qquad u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{in } \mathbb{R}^n. \tag{3.1}$$

The solution is represented by the potentials associated with the Gaussian kernel $K(\mathbf{x},t) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$:

$$u(\mathbf{x},t) = K_t * u_0(\mathbf{x}) = \int_{\mathbb{R}^n} K(\mathbf{x}-\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y}. \tag{3.2}$$

By the series representation of the Gaussian kernel, $u(\mathbf{x},t)$ could be expressed by

$$u(\mathbf{x},t) = \sum_{\alpha\geqslant 0}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K(\mathbf{x},t)\int_{\mathbb{R}^n} u_0(\mathbf{y})\mathbf{y}^{\alpha}\,d\mathbf{y}. \tag{3.3}$$

In fact, the first identity

$$\int_{\mathbb{R}^n} K(\mathbf{x}-\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y} = \sum_{\alpha\geqslant 0}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K(\mathbf{x},t)\int_{\mathbb{R}^n} u_0(\mathbf{y})y^{\alpha}\,d\mathbf{y}$$

holds for sufficiently rapid decaying u_0 such as a function in the Schwarz class. For general u_0 this holds no more true. For example it might be $u_0\mathbf{y}^{\beta} \notin L^1$ for some β. Therefore, it is necessary to apply a kind of Taylor theorem to represent $K(\mathbf{x}-\mathbf{y},t)$ as

$$K(\mathbf{x}-\mathbf{y},t) = \sum_{0\leqslant|\alpha|\leqslant m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^{\alpha}K(\mathbf{x},t) + R_m(\mathbf{x},\mathbf{y},t),$$

where the remainder term is

$$R_m(\mathbf{x},\mathbf{y},t) = \sum_{|\alpha|=m+1}\int_0^1 \frac{(-1)^{m+1}}{\alpha!}\left(\partial_{\mathbf{x}}^{\alpha}K\right)(\mathbf{x}-c\mathbf{y},t)\mathbf{y}^{\alpha}\,dc.$$

Hence, if $u_0\mathbf{y}^\alpha \in L^1$ for all $|\alpha| \leqslant m+1$, we have

$$\int_{\mathbb{R}^n} K(\mathbf{x}-\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y} = \sum_{0\leqslant|\alpha|\leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^\alpha K(\mathbf{x},t)\int_{\mathbb{R}^n} u_0(\mathbf{y})\mathbf{y}^\alpha\,d\mathbf{y} + \int_{\mathbb{R}^n} u_0(\mathbf{y})R_m(\mathbf{x},\mathbf{y},t)\,d\mathbf{y}.$$

Straight forward computation yields

$$\left|R_m(\mathbf{x},\mathbf{y},t)\right| \leqslant c(m)t^{-\frac{m+1}{2}-\frac{n}{2}}|\mathbf{y}|^{m+1}\int_0^1 e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,dc.$$

From this estimate of R_m we have

$$\left|\int_{\mathbb{R}^n} u_0(\mathbf{y})R_m(\mathbf{x},\mathbf{y},t)\,d\mathbf{y}\right| \leqslant C(m)t^{-\frac{m+1}{2}-\frac{n}{2}}\int_0^1\int_{\mathbb{R}^n}\left|u_0(\mathbf{y})\right||\mathbf{y}|^{m+1}e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,d\mathbf{y}\,dc.$$

Theorem 3.1. *Let $u_0y^\alpha \in L^1$ for all $|\alpha| \leqslant m+1$, and $|\mathbf{y}|^{m+1+l}u_0(\mathbf{y}) \in L^1$ for some l. Then the heat solution $u(\mathbf{x},t) = (K_t * u_0)(\mathbf{x})$ satisfies*

$$\left\| |\cdot|^l\left[(K_t*u_0)(\cdot) - \sum_{0\leqslant|\alpha|\leqslant m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^\alpha K(\cdot,t)\int_{\mathbb{R}^n} u_0(\mathbf{y})y^\alpha\,d\mathbf{y}\right]\right\|_{L^p} = O\left(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{p})}\right)$$

as t goes to the infinity, for $1 \leqslant p \leqslant \infty$.

Proof. From the estimate of R_m

$$\left|R_m(\mathbf{x},\mathbf{y},t)\right| \leqslant C(m)t^{-\frac{m+1}{2}-\frac{n}{2}}|\mathbf{y}|^{m+1}\int_0^1 e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,dc,$$

we have that

$$|\mathbf{x}|^l\left|R_m(\mathbf{x},\mathbf{y},t)\right| \leqslant C(m)\left[t^{-\frac{m+1}{2}-\frac{n}{2}}|\mathbf{y}|^{m+1}\int_0^1|\mathbf{x}-c\mathbf{y}|^l e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,dc + C(m)t^{-\frac{m+1}{2}-\frac{n}{2}}|\mathbf{y}|^{m+1+l}\int_0^1 e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,dc\right].$$

Note that

$$|\mathbf{x}|^l \leqslant |\mathbf{x}-c\mathbf{y}|^l + |\mathbf{y}|^l,$$

and

$$|\mathbf{x}-c\mathbf{y}|^l e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}} \leqslant C(l)t^{\frac{l}{2}}e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{16t}}.$$

Hence we have that

$$\begin{aligned}
&|\mathbf{x}|^l\left[(K_t*u_0)(\mathbf{x}) - \sum_{0\leqslant|\alpha|\leqslant m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^\alpha K(\mathbf{x},t)\int_{\Omega} u_0(\mathbf{y})y^\alpha\,d\mathbf{y}\right] \\
&\quad\leqslant C(m)t^{-\frac{m+1}{2}-\frac{n}{2}+\frac{l}{2}}\int_0^1\int |\mathbf{y}|^{m+1}\left|u_0(\mathbf{y})\right|e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,d\mathbf{y}\,dc \\
&\qquad + C(m)t^{-\frac{m+1}{2}-\frac{n}{2}}\int_0^1\int |\mathbf{y}|^{m+1+l}\left|u_0(\mathbf{y})\right|e^{-\frac{|\mathbf{x}-c\mathbf{y}|^2}{8t}}\,d\mathbf{y}\,dc \\
&\quad= I + II.
\end{aligned}$$

We note that

$$I = C(m)t^{-\frac{m+1}{2}+\frac{l}{2}} \int_0^1 u_{0,1,c} * K_{4t}(\mathbf{x})\,dc, \qquad II = C(m)t^{-\frac{m+1}{2}} \int_0^1 u_{0,2,c} * K_{2t}(\mathbf{x})\,dc,$$

where $u_{0,1,c}(\mathbf{z}) = |u_0(\frac{\mathbf{z}}{c})||\mathbf{z}|^{m+1}c^{-m-1-n}$ and $u_{0,2,c}(\mathbf{z}) = |u_0(\frac{\mathbf{z}}{c})||\mathbf{z}|^{m+1+l}c^{-m-1-l-n}$. By Young's inequality we have

$$\left\|u_{0,1,c} * K_{4t}(\mathbf{x})\right\|_{L^p} \leqslant \|u_{0,1,c}\|_{L^1} t^{-\frac{n}{2}(1-\frac{1}{p})}, \qquad \left\|u_{0,2,c} * K_{2t}(\mathbf{x})\right\|_{L^p} \leqslant \|u_{0,2,c}\|_{L^1} t^{-\frac{n}{2}(1-\frac{1}{p})}.$$

With the obvious integration, it holds

$$\|u_{0,1,c}\|_{L^1} = \left\||\mathbf{y}|^{m+1}u_0(\mathbf{y})\right\|_{L^1}, \qquad \|u_{0,2,c}\|_{L^1} = \left\||\mathbf{y}|^{m+1+l}u_0(\mathbf{y})\right\|_{L^1}.$$

Hence we have that

$$I \leqslant c(m)t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{p})}\left\||\mathbf{y}|^{m+1}u_0(\mathbf{y})\right\|_{L^1},$$
$$II \leqslant c(m)t^{-\frac{m+1}{2}-\frac{n}{2}(1-\frac{1}{p})}\left\||\mathbf{y}|^{m+1+l}u_0(\mathbf{y})\right\|_{L^1}.$$

This leads to the proof of our theorem for the case of $1 \leqslant p \leqslant \infty$. □

Now, let us consider the Stokes equations:

$$\operatorname{div}\mathbf{v} = 0, \qquad \partial_t \mathbf{v} - \Delta\mathbf{v} + \nabla q = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \tag{3.4}$$

with the initial condition

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^n.$$

Then the solution of the Stokes equation is represented by the Gaussian kernel as follows:

$$\mathbf{v}(\mathbf{x}, t) = K_t * \mathbf{u}_0(\mathbf{x}) = \int_{\mathbb{R}^n} K(\mathbf{x}-\mathbf{y}, t)\mathbf{u}_0(\mathbf{y})\,d\mathbf{y}.$$

By Theorem 3.1 we derive the following asymptotic behavior for the Stokes solution.

Theorem 3.2. *Suppose* $\operatorname{div}\mathbf{u}_0 = 0$. *We let* $\mathbf{y}^\alpha\mathbf{u}_0 \in L^1$ *for all* $|\alpha| \leqslant m+1$ *and* $\mathbf{u}_0|\mathbf{y}|^{m+1+l} \in L^1$ *for some* $l \geqslant 0$. *Then the Stokes solution* $\mathbf{v} = K_t * \mathbf{u}_0$ *satisfies that for* $1 \leqslant q \leqslant \infty$

$$\left\||\cdot|^l\left[K_t * \mathbf{u}_0(\cdot) - \sum_{1\leqslant|\alpha|\leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\mathbf{x}}^\alpha K(\cdot,t)\int \mathbf{u}_0(\mathbf{y})\mathbf{y}^\alpha\,d\mathbf{y}\right]\right\|_{L^q} = O\left(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\right).$$

Here we used the fact that $\operatorname{div}\mathbf{u}_0 = 0$ and $\mathbf{u}_0 \in L^1$ implies that $\int_{\mathbb{R}^n}\mathbf{u}_0(\mathbf{y})\,d\mathbf{y} = 0$. (See the paper of T. Miyakawa [4].)

From Theorem 3.2 we have the following reverse estimate of the Stokes solutions corresponding to special class of initial data.

Corollary 3.3. *Suppose* $\mathbf{u}_0$ *satisfies the same assumption in Theorem* 3.2. *We add one more assumption that*

$$\int_{\mathbb{R}^n} \mathbf{y}^\alpha\mathbf{u}_0(\mathbf{y})\,d\mathbf{y} = 0 \quad \textit{for all } |\alpha| < m \quad \textit{and} \quad \int_{\mathbb{R}^n} \mathbf{y}^\alpha\mathbf{u}_0(\mathbf{y})\,d\mathbf{y} \neq 0 \quad \textit{for some } |\alpha| = m.$$

Then $\mathbf{u}$ *satisfies that*

$$\left\||\cdot|^l K_t * \mathbf{u}_0(\cdot)\right\|_{L^q} \geqslant Ct^{-\frac{m}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}$$

for large $t > 0$.

4. Proof of Theorem 1.1

Let us consider a solution of the Navier–Stokes equations (1.1) with the initial velocity $\mathbf{u}_0(\mathbf{x})$.

A solution of the Navier–Stokes equations is expressed by the potentials with the Gaussian kernel:

$$u_i(\mathbf{x},t) = K_t * u_{0,i}(\mathbf{x}) + \int_0^t ds \int_{\mathbb{R}^n} (u_d u_j)(\mathbf{y},s) F^i_{dj}(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y}.$$

The assumption that $\operatorname{div}\mathbf{u}_0 = 0$, $\mathbf{y}^\alpha \mathbf{u}_0 \in L^1$ for $|\alpha| \leqslant m+1$, and $|\mathbf{y}|^{m+l+1}\mathbf{u}_0 \in L^1$ implies the following estimate of Theorem 3.2: for $1 \leqslant q \leqslant \infty$

$$\left\| |\cdot|^l \left[K_t * u_{0,i}(\cdot) - \sum_{0\leqslant|\alpha|\leqslant m} \frac{(-1)^n}{\alpha!} \partial^\alpha_{\mathbf{x}} K(\cdot,t) \int_{\mathbb{R}^n} u_{0,i}(\mathbf{y})\mathbf{y}^\alpha\,d\mathbf{y} \right]\right\|_{L^q} = O\bigl(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\bigr), \quad 1 \leqslant q \leqslant \infty.$$

Therefore we have only to prove that

$$\begin{aligned}
&\Biggl\| |\cdot|^l \Biggl[\int_0^t ds \int_{\mathbb{R}^n} (u_d u_j)(\mathbf{y},s) F^i_{dj}(\cdot-\mathbf{y},t-s)\,d\mathbf{y} \\
&\quad - \sum_{0\leqslant|\alpha|+2p\leqslant m-1} \frac{(-1)^{|\alpha|+2p}}{\alpha!p!} \bigl(\partial^\alpha_x \partial^p_t F^i_{dj}\bigr)(\cdot,t) \int_0^\infty \int (u_d u_j)(\mathbf{y},s)\mathbf{y}^\alpha s^p\,d\mathbf{y}\,ds \Biggr]\Biggr\|_q \\
&= \begin{cases} O(t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}), & \text{if } m < n, \\ O(t^{-\frac{n+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\log t), & \text{if } m = n, \end{cases}
\end{aligned}$$

for $1 \leqslant q \leqslant \infty$ and $0 \leqslant l < n(1-\frac{1}{q})+1$.

Note that

$$\begin{aligned}
&u_i(\mathbf{x},t) - \int K(\mathbf{x}-\mathbf{y},t)u_{0,i}(\mathbf{y})\,d\mathbf{y} \\
&= \int_0^t ds \int (u_d u_j)(\mathbf{y},s) F^i_{dj}(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y} \\
&= \int_{t/2}^t ds \int (u_d u_j)(\mathbf{y},s) F^i_{dj}(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y} + \int_0^{t/2} \int (u_d u_j)(\mathbf{y},s) F^i_{dj}(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y} \\
&= J_1 + J_2.
\end{aligned}$$

Recalling (2.1), we have

$$\bigl|F^i_{dj}(\mathbf{x}-\mathbf{y},t-s)\bigr| \leqslant (t-s)^{-\frac{n+1}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}} + \int_{t-s}^\infty \tau^{-\frac{n+3}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}}\,d\tau,$$

and thus

$$J_1 \leqslant \int_{t/2}^t (t-s)^{-\frac{n+1}{2}} \int \bigl|\mathbf{u}(\mathbf{y},s)\bigr|^2 e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}}\,d\mathbf{y}\,ds + \int_{t/2}^t \int_{t-s}^\infty \tau^{-\frac{n+3}{2}} \int \bigl|\mathbf{u}(\mathbf{y},s)\bigr|^2 e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}}\,d\mathbf{y}\,d\tau\,ds = J_{11} + J_{12}.$$

Since $|\mathbf{x}|^l \leqslant |\mathbf{x}-\mathbf{y}|^l + |\mathbf{y}|^l$ and

$$|\mathbf{x}-\mathbf{y}|^l e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}} \leqslant C(l)(t-s)^{\frac{l}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{8(t-s)}},$$

we have

$$J_{11}(\mathbf{x},t)|\mathbf{x}|^l \leqslant \int_{t/2}^{t} (t-s)^{-\frac{n+1}{2}+\frac{l}{2}} \int \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{8(t-s)}}\,d\mathbf{y}\,ds + \int_{t/2}^{t} (t-s)^{-\frac{n+1}{2}} \int \left|\mathbf{u}(\mathbf{y},s)\right|^2 |\mathbf{y}|^l e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}}\,d\mathbf{y}\,ds.$$

Note that

$$\begin{aligned}(t-s)^{-\frac{n+1}{2}+\frac{l}{2}} \left\| \int \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{8(t-s)}}\,d\mathbf{y}\right\|_q &= (t-s)^{\frac{l-1}{2}} \left\| K_{\sqrt{2}(t-s)} * \left|\mathbf{u}(\cdot,s)\right|^2\right\|_{L^q} \\ &\leqslant C(t-s)^{\frac{l-1}{2}} \left\|\mathbf{u}(\cdot,s)\right\|_{L^{2q}}^2 \\ &\leqslant (1+s)^{-1-n(1-\frac{1}{2q})}(t-s)^{-\frac{1}{2}+\frac{l}{2}},\end{aligned}$$

and

$$\begin{aligned}(t-s)^{-\frac{n+1}{2}} \left\| \int |\mathbf{y}|^l \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{8(t-s)}}\,d\mathbf{y}\right\|_q &= (t-s)^{-\frac{1}{2}} \left\| K_{\sqrt{2}(t-s)} * \left(|\cdot|^l \left|\mathbf{u}(\cdot,s)\right|^2\right)\right\|_{L^q} \\ &\leqslant c(t-s)^{-\frac{1}{2}} \left\||\cdot|^{\frac{l}{2}}\mathbf{u}(\cdot,s)\right\|_{2q}^2 \\ &\leqslant c(t-s)^{-\frac{1}{2}} \left\||\cdot|^{n+1}\mathbf{u}(\cdot,s)\right\|_{L^\infty}^{\frac{l}{n+1}} \left\|\mathbf{u}(\cdot,s)\right\|_{\frac{2(n+1)-l}{n+1}q}^{\frac{2(n+1)-l}{n+1}} \\ &\leqslant (1+s)^{-1+\frac{l}{2}-n+\frac{n}{2q}}(t-s)^{-\frac{1}{2}}.\end{aligned}$$

Here we used the fact that $\mathbf{u}$ satisfies the properties (1.2), (1.3).

Hence, we have

$$\begin{aligned}\left\|J_{11}(\cdot,t)|\cdot|^l\right\|_q &\leqslant \int_{t/2}^{t} (t-s)^{-\frac{n+1}{2}+\frac{l}{2}} \left\|\int \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{8(t-s)}}\,d\mathbf{y}\right\|_q ds \\ &\quad + \int_{t/2}^{t} (t-s)^{-\frac{n+1}{2}} \left\|\int \left|\mathbf{u}(\mathbf{y},s)\right|^2 |\mathbf{y}|^l e^{-\frac{|\cdot-\mathbf{y}|^2}{8(t-s)}}\,d\mathbf{y}\right\|_q ds \\ &\leqslant \int_{t/2}^{t} (1+s)^{-1-n(1-\frac{1}{2q})}(t-s)^{-\frac{1}{2}+\frac{l}{2}}\,ds + \int_{t/2}^{t} (1+s)^{-1+\frac{l}{2}-n+\frac{n}{2q}}(t-s)^{-\frac{1}{2}}\,ds \\ &\leqslant (1+t)^{-\frac{n+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}.\end{aligned}$$

This implies that

$$\left\|J_{11}(\cdot,t)|\cdot|^l\right\|_q = O\left((1+t)^{-\frac{1+n}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\right). \tag{4.1}$$

Now we would derive the weighted estimates for J_{12}. Noting that $|\mathbf{x}|^l \leqslant |\mathbf{x}-\mathbf{y}|^l + |\mathbf{y}|^l$ and $|\mathbf{x}-\mathbf{y}|^l e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\tau}} \leqslant C(l)\tau^{\frac{l}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{8\tau}}$, we have

$$\begin{aligned}\left\||\cdot|^l J_{12}(\cdot,t)\right\|_q &\leqslant \int_{t/2}^{t}\int_{t-s}^{\infty} \tau^{-\frac{n+3}{2}+\frac{l}{2}} \left\|\int \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{8\tau}}\,d\mathbf{y}\right\|_q d\tau\,ds \\ &\quad + \int_{t/2}^{t}\int_{t-s}^{\infty} \tau^{-\frac{n+3}{2}} \left\|\int |\mathbf{y}|^l \left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{4\tau}}\,d\mathbf{y}\right\|_q d\tau\,ds.\end{aligned}$$

We note that

$$\begin{aligned}\tau^{-\frac{n+3}{2}+\frac{l}{2}}\left\|\int\left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{8\tau}}\,d\mathbf{y}\right\|_q &= \tau^{\frac{l-3}{2}}\left\|K_{\sqrt{2}\tau}*\left|\mathbf{u}(\cdot,s)\right|^2\right\|_{L^q}\\ &\leqslant C\tau^{\frac{l-3}{2}}\|K_{\sqrt{2}\tau}\|_{L^q}\left\|\mathbf{u}(\cdot,s)\right\|_{L^2}^2\\ &\leqslant C(1+s)^{-1-\frac{n}{2}}\tau^{\frac{l-3}{2}-\frac{n}{2}(1-\frac{1}{q})},\end{aligned}$$

and

$$\begin{aligned}\tau^{-\frac{n+3}{2}}\left\|\int|\mathbf{y}|^l\left|\mathbf{u}(\mathbf{y},s)\right|^2 e^{-\frac{|\cdot-\mathbf{y}|^2}{4\tau}}\,d\mathbf{y}\right\|_q &= \tau^{-\frac{3}{2}}\left\|K_{\sqrt{2}(t-s)}*\left(|\cdot|^l\left|\mathbf{u}(\cdot,s)\right|^2\right)\right\|_{L^q}\\ &\leqslant C\tau^{-\frac{3}{2}}\left\||\cdot|^l\left|\mathbf{u}(\cdot,s)\right|^2\right\|_q\\ &\leqslant \tau^{-\frac{3}{2}}\left\||\cdot|^{n+1}\mathbf{u}(\cdot,s)\right\|_{L^\infty}^{\frac{l}{n+1}}\left\|\mathbf{u}(\cdot,s)\right\|_{\frac{2(n+1)-l}{n+1}q}^{\frac{2(n+1)-l}{n+1}}\\ &\leqslant C(1+s)^{-1+\frac{l}{2}-n+\frac{n}{2q}}\tau^{-\frac{3}{2}}.\end{aligned}$$

Hence we have

$$\begin{aligned}\left\|J_{12}(\cdot,t)\right\|_q &\leqslant \int_{t/2}^{t}\int_{t-s}^{\infty}\tau^{-\frac{3}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}(1+s)^{-1-\frac{n}{2}}\,d\tau\,ds\\ &\leqslant \int_{t/2}^{t}(1+s)^{-1-\frac{n}{2}}(t-s)^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,ds\\ &\leqslant C(1+t)^{-\frac{1+n}{2}-\frac{n}{2}(1-\frac{1}{q})},\end{aligned}$$

and

$$\begin{aligned}\left\||\cdot|^l J_{12}(\cdot,t)\right\|_q &\leqslant \int_{t/2}^{t}\int_{t-s}^{\infty}\tau^{-\frac{3}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}(1+s)^{-1-\frac{n}{2}}\,d\tau\,ds+\int_{t/2}^{t}\int_{t-s}^{\infty}\tau^{-\frac{3}{2}}(1+s)^{-1+\frac{l}{2}-n+\frac{n}{2q}}\,d\tau\,ds\\ &\leqslant \int_{t/2}^{t}(1+s)^{-1-\frac{n}{2}}(t-s)^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,ds+\int_{t/2}^{t}(1+s)^{-1+\frac{l}{2}-n+\frac{n}{2q}}(t-s)^{-\frac{1}{2}}\,ds\\ &\leqslant C(1+t)^{-\frac{1+n}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})},\end{aligned}$$

when $n(1-\frac{1}{q})-1<l<n(1-\frac{1}{q})+1$. Here we noted that

$$\int_{t-s}^{\infty}\tau^{-\frac{3}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,d\tau=(t-s)^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\quad\text{for } l<n\left(1-\frac{1}{q}\right)+1$$

and

$$\int_{t/2}^{t}(t-s)^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,ds=Ct^{\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\quad\text{for } l>n\left(1-\frac{1}{q}\right)-1.$$

By the interpolation property this implies that

$$\left\||\cdot|^l J_{12}(\cdot,t)\right\|_q \leqslant (t-s)^{-\frac{1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\quad\text{for any } 0\leqslant l<n\left(1-\frac{1}{q}\right)+1. \tag{4.2}$$

From Lemma 2.3 J_2 is decomposed as

$$
\begin{aligned}
J_2 = {} & \sum_{0\leqslant|\alpha|+2p\leqslant m-1} \frac{(-1)^{|\alpha|+2p}}{\alpha!p!}\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\int_0^{t/2}\int (u_d u_j)(\mathbf{y},s)\mathbf{y}^\alpha s^p\,d\mathbf{y}\,ds \\
& + \int_0^{t/2}\int (u_j u_p)(\mathbf{y},s)R_{(m-1)dji}(x,y,t,s)\,d\mathbf{y}\,ds \\
= {} & \sum_{0\leqslant|\alpha|+2p\leqslant m-1} \frac{(-1)^{|\alpha|+2p}}{\alpha!p!}\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\int_0^{\infty}\int (u_d u_j)(\mathbf{y},s)\mathbf{y}^\alpha s^p\,d\mathbf{y}\,ds \\
& - \sum_{0\leqslant|\alpha|+2p\leqslant m-1} \frac{(-1)^{|\alpha|+2p}}{\alpha!p!}\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\int_{t/2}^{\infty}\int (u_d u_j)(\mathbf{y},s)\mathbf{y}^\alpha s^p\,d\mathbf{y}\,ds \\
& + \int_0^{t/2}\int (u_d u_j)(\mathbf{y},s)R_{(m-1)dji}(x,y,t,s)\,d\mathbf{y}\,ds \\
= {} & J_{21}+J_{22}+J_{23}.
\end{aligned}
$$

Since for $|\alpha|+2p=k$

$$
\big|\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\big| \leqslant t^{-\frac{k}{2}-\frac{n+1}{2}}e^{-\frac{|\mathbf{x}|^2}{4t}} + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}}e^{-\frac{|\mathbf{x}|^2}{4\tau}}\,d\tau,
$$

we have

$$
\begin{aligned}
|\mathbf{x}|^l\big|\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\big| &\leqslant t^{-\frac{k}{2}-\frac{n+1}{2}}|\mathbf{x}|^l e^{-\frac{|\mathbf{x}|^2}{4t}} + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}}|\mathbf{x}|^l e^{-\frac{|\mathbf{x}|^2}{4\tau}}\,d\tau \\
&\leqslant t^{-\frac{k}{2}-\frac{n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8t}} + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8\tau}}\,d\tau.
\end{aligned}
$$

We also note that

$$
\left(\int |\mathbf{y}|^\alpha |\mathbf{u}(\cdot,s)|^2\,d\mathbf{y}\right)s^p \leqslant (1+s)^{\frac{|\alpha|}{2}-1-\frac{n}{2}}s^p \leqslant s^{\frac{k}{2}-1-\frac{n}{2}} \quad \text{for } |\alpha|+2p=k.
$$

Hence we have that

$$
\begin{aligned}
|\mathbf{x}|^l J_{22}(\mathbf{x},t) &\leqslant \sum_{0\leqslant|\alpha|+2p\leqslant m-1}\frac{1}{\alpha!p!}|\mathbf{x}|^l\big|\big(\partial_x^\alpha \partial_t^p F_{dj}^i\big)(\mathbf{x},t)\big|\int_{t/2}^\infty\left(\int |\mathbf{y}|^\alpha\big|\mathbf{u}(\cdot,s)\big|^2\,d\mathbf{y}\right)s^p\,ds \\
&\leqslant \sum_{k=0}^{m-1}\left[t^{-\frac{k}{2}-\frac{n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8t}} + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8\tau}}\,d\tau\right]\int_{t/2}^\infty (1+s)^{\frac{k}{2}-1-\frac{n}{2}}\,ds \\
&\leqslant \sum_{k=0}^{m-1}\left[t^{-\frac{k}{2}-\frac{n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8t}} + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}|^2}{8\tau}}\,d\tau\right]t^{\frac{k}{2}-\frac{n}{2}}.
\end{aligned}
$$

Here we note that the integral $\int_{t/2}^\infty (1+s)^{\frac{k}{2}-1-\frac{n}{2}}\,ds = C(1+t)^{\frac{k}{2}-\frac{n}{2}}$ holds for $k<n$. (At this point we need the restriction $m\leqslant n$.) Hence we have that

$$\begin{aligned}\big\| |\cdot|^l J_{22}(\cdot,t)\big\|_q &\leqslant \sum_{k=0}^{m-1}\left[t^{\frac{n-1}{2}-\frac{k}{2}-\frac{n+1}{2}}\big\|e^{-\frac{|\mathbf{x}|^2}{8t}}\big\|_q + \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}}\big\|e^{-\frac{|\mathbf{x}|^2}{8\tau}}\big\|_q\,d\tau\right]t^{\frac{k}{2}-\frac{n}{2}} \\ &\leqslant C(m)t^{\frac{l}{2}-\frac{n+1}{2}-\frac{n}{2}(1-\frac{1}{q})}, \quad \text{when } l < n\left(1-\frac{1}{q}\right)+1+k.\end{aligned}$$

Here we noted that

$$\int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}}\big\|e^{-\frac{|\mathbf{x}|^2}{8\tau}}\big\|_q\,d\tau = \int_t^\infty \tau^{-\frac{k}{2}-\frac{n+3}{2}+\frac{l}{2}+\frac{n}{2q}}\,d\tau = Ct^{-\frac{k}{2}+\frac{l}{2}-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}, \quad \text{when } l < n\left(1-\frac{1}{q}\right)+1+k.$$

This implies that

$$\big\| |\cdot|^l J_{22}(\cdot,t)\big\|_q = O\big(t^{\frac{l}{2}-\frac{n+1}{2}-\frac{n}{2}(1-\frac{1}{q})}\big), \quad \text{when } l < n\left(1-\frac{1}{q}\right)+1. \tag{4.3}$$

Finally we consider J_{23}. Since $R_{(m-1)idj}$ is bounded by

$$\begin{aligned}&R_{(m-1)idj}(\mathbf{x},\mathbf{y},t,s) \\ &\quad\leqslant C\int_0^1\int_0^1\left[(t-c_2s)^{-\frac{m+n+1}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8(t-c_2s)}} + \int_{t-c_2s}^\infty \tau^{-\frac{m+n+3}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\tau\right]dc_1\,dc_2\cdot\sum_{|\alpha|+2p=m}|\mathbf{y}|^\alpha s^p,\end{aligned}$$

J_{23} is bounded by

$$\begin{aligned}J_{23} \leqslant \sum_{|\alpha|+2p=m}\int_0^1\int_0^1\int_0^{t/2}\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|}s^p\Bigg[&(t-c_2s)^{-\frac{m+n+1}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8(t-c_2s)}} \\ &+ \int_{t-c_2s}^\infty \tau^{-\frac{m+n+3}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\tau\Bigg]d\mathbf{y}\,ds\,dc_1\,dc_2.\end{aligned}$$

Note that $|\mathbf{x}|^l \leqslant |\mathbf{x}-c_1\mathbf{y}|^l + |\mathbf{y}|^l$ for $0 \leqslant c_1 \leqslant 1$,

$$|\mathbf{x}-c_1\mathbf{y}|^l e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8(t-c_2s)}} \leqslant C(l)(t-c_2s)^{\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16(t-c_2s)}},$$

that for $0 \leqslant s \leqslant \frac{t}{2}$

$$(t-c_2s)^{-\frac{m+n+1}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8(t-c_2s)}} + \int_{t-c_2s}^\infty \tau^{-\frac{m+n+3}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\tau \leqslant \left(\frac{t}{2}\right)^{-\frac{m+n+1}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8t}} + \int_{t/2}^\infty \tau^{-\frac{m+n+3}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\tau,$$

$$\begin{aligned}&(t-c_2s)^{-\frac{m+n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16(t-c_2s)}} + \int_{t-c_2s}^\infty \tau^{-\frac{m+n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16\tau}}\,d\tau \\ &\quad\leqslant \left(\frac{t}{2}\right)^{-\frac{m+n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16t}} + \int_{t/2}^\infty \tau^{-\frac{m+n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16\tau}}\,d\tau,\end{aligned}$$

we have

$$\begin{aligned}|\mathbf{x}|^l J_{23}(\mathbf{x},t) \leqslant{}& \int_0^1\int_0^{t/2}s^p\,ds\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|}\left[t^{-\frac{m+n+1}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16t}} + \int_{t/2}^\infty \tau^{-\frac{m+n+3}{2}+\frac{l}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16\tau}}\,d\tau\right]d\mathbf{y}\,dc_1 \\ &+ \int_0^1\int_0^{t/2}s^p\,ds\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}\left[t^{-\frac{m+n+1}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8t}} + \int_{t/2}^\infty \tau^{-\frac{m+n+3}{2}}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\tau\right]d\mathbf{y}\,dc_1.\end{aligned}$$

Hence we have

$$\begin{aligned}\big\||\cdot|^l J_{23}(\cdot,t)\big\|_q \leqslant \int_0^1\int_0^{t/2} s^p\Bigg[t^{-\frac{m+n+1}{2}+\frac{l}{2}}\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16t}}\,d\mathbf{y}\bigg\|_q \\ +\int_{t/2}^{\infty}\tau^{-\frac{m+n+3}{2}+\frac{l}{2}}\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|}e^{-\frac{|\cdot-c_1\mathbf{y}|^2}{16\tau}}\,d\mathbf{y}\bigg\|_q\,d\tau\Bigg]\,ds\,dc_1 \\ +\int_0^1\int_0^{t/2} s^p\Bigg[t^{-\frac{m+n+1}{2}}\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}e^{-\frac{|\cdot-c_1\mathbf{y}|^2}{8t}}\bigg\|_q \\ +\int_{t/2}^{\infty}\tau^{-\frac{m+n+3}{2}}\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8\tau}}\,d\mathbf{y}\bigg\|_q\,d\tau\Bigg]\,ds\,dc_1.\end{aligned}$$

Note that

$$\begin{aligned}\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{\alpha}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{16t}}\,d\mathbf{y} &= (2t)^{\frac{n}{2}}\big(\mathbf{u}_{1,\alpha,c_1}(s)*K_{4t}\big)(\mathbf{x}),\\ \int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}e^{-\frac{|\mathbf{x}-c_1\mathbf{y}|^2}{8t}}\,d\mathbf{y} &= t^{\frac{n}{2}}\big(\mathbf{u}_{2,\alpha,c_1}(s)*K_{2t}\big)(\mathbf{x}),\end{aligned}$$

where

$$\mathbf{u}_{1,\alpha,c_1}(s)(\mathbf{x}) = c_1^{-n-|\alpha|}\left|\mathbf{u}\left(\frac{\mathbf{x}}{c_1},s\right)\right|^2|\mathbf{x}|^{\alpha},\qquad \mathbf{u}_{2,\alpha,c_1}(s)(\mathbf{x}) = c_1^{-n-|\alpha|-l}\left|\mathbf{u}\left(\frac{\mathbf{x}}{c_1},s\right)\right|^2|\mathbf{x}|^{|\alpha|+l}.$$

We also note that

$$\big\|\mathbf{u}_{1,\alpha,c_1}(\cdot,s)\big\|_{L^1} = \int\big|\mathbf{u}(\mathbf{x},s)\big|^2|\mathbf{x}|^{\alpha}\,d\mathbf{x},\qquad \big\|\mathbf{u}_{2,\alpha,c_1}(\cdot,s)\big\|_{L^1} = \int\big|\mathbf{u}(\mathbf{x},s)\big|^2|\mathbf{x}|^{|\alpha|+l}\,d\mathbf{x}.$$

By the hypothesis

$$\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{\alpha}\,d\mathbf{y} \leqslant (1+s)^{\frac{|\alpha|}{2}-1-\frac{n}{2}},\qquad \int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}\,d\mathbf{y} \leqslant (1+s)^{\frac{|\alpha|+l}{2}-1-\frac{n}{2}}.$$

Hence we have that for $|\alpha|+2p=m$

$$\begin{aligned}s^p\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{\alpha}e^{-\frac{|\cdot-c_1y|^2}{16t}}\,d\mathbf{y}\bigg\|_q &\leqslant s^p t^{\frac{n}{2q}}\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|}\,d\mathbf{y} \leqslant Ct^{\frac{n}{2q}}(1+s)^{-1-\frac{n}{2}+\frac{m}{2}},\\ s^p\bigg\|\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}e^{-\frac{|\cdot-c_1\mathbf{y}|^2}{8t}}\,d\mathbf{y}\bigg\|_q &\leqslant Cs^p t^{\frac{n}{2q}}\int\big|\mathbf{u}(\mathbf{y},s)\big|^2|\mathbf{y}|^{|\alpha|+l}\,d\mathbf{x} \leqslant t^{\frac{n}{2q}}(1+s)^{-1+\frac{l}{2}-\frac{n}{2}+\frac{m}{2}}.\end{aligned}$$

This implies that

$$\begin{aligned}\big\||\cdot|^l J_{23}(\cdot,t)\big\|_q \leqslant \left[t^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}+\int_{t/2}^{\infty}\tau^{-\frac{m+3}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,d\tau\right]\int_0^{t/2}(1+s)^{-1-\frac{n}{2}+\frac{m}{2}}\,ds \\ +\left[t^{-\frac{m+1}{2}-\frac{n}{2}(1-\frac{1}{q})}+\int_{t/2}^{\infty}\tau^{-\frac{m+3}{2}-\frac{n}{2}(1-\frac{1}{q})}\,d\tau\right]\int_0^{t/2}(1+s)^{-1+\frac{l}{2}-\frac{n}{2}+\frac{m}{2}}\,ds\end{aligned}$$

$$\leqslant \begin{cases} Ct^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}, & \text{if } m < n, \\ Ct^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\log(t+1) + Ct^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}, & \text{if } m = n. \end{cases}$$

Here we noted that $\int_{t/2}^{\infty} \tau^{-\frac{m+3}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}\,d\tau = Ct^{-\frac{m+1}{2}+\frac{l}{2}-\frac{n}{2}(1-\frac{1}{q})}$ for $l < m+1+n(1-\frac{1}{q})$.

This completes the proof of Theorem 1.1.

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